

# Spin-3 quasinormal modes of BTZ black hole

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## Abstract

Using the operator approach, we obtain quasinormal modes (QNMs) of BTZ black hole in spin-3 topologically massive gravity by solving the first-order equation of motion with the transverse-traceless condition. We find that these are different from those obtained when solving the second-order differential equation for the third-rank tensor of spin-3 field subject to suitable boundary conditions and having the sign ambiguity of mass. However, it is shown clearly that two approaches to the left-moving QNMs are identical, while the right-moving QNMs of solving the second-order equation are given by descendants of the operator approach.

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# 1 Introduction

Recently, higher-spin theories on  $\text{AdS}_3$  have been paid much attention because they admit a truncation to an arbitrary maximal spin  $N$  [1, 2]. Especially, the prototype of spin-3 model is a third-rank tensor of spin-3 field coupled to topologically massive gravity. The authors [3] have discussed the traceless spin-3 fluctuations around  $\text{AdS}_3$  spacetimes and, found that there exists a single massive propagating mode, besides left-moving and right-moving massless modes (gauge artifacts). Also, a trace part of spin-3 fluctuations on  $\text{AdS}_3$  spacetimes has been studied in Ref. [4]. However, such a massive trace mode has zero energy and becomes pure gauge at the chiral point. These are considered through extended analysis of spin-2 field in the cosmological topologically massive gravity [5].

Very recently, Datta and David [6] have solved massive wave equations of arbitrary integer spin fields including spin-3 fields in the BTZ black hole background, and have obtained their quasinormal modes which are consistent with the location of the poles of the corresponding two-point function in the dual conformal field theory. This could be predicted by the  $\text{AdS}_3/\text{CFT}_2$  correspondence. They have considered the second-order equation of  $[\bar{\square} - m^2 + 4/\ell^2]\Phi_{\rho\mu\nu} = 0$  for spin-3 fields with the ingoing modes at horizon and Dirichlet boundary condition at infinity. However, in this case, one confronts with sign ambiguity of mass  $m$ . Thus, in order to avoid this ambiguity, one could solve the first-order equation of  $\epsilon_\rho^{\alpha\beta}\bar{\nabla}_\alpha\Phi_{\beta\mu\nu} + m\Phi_{\rho\mu\nu} = 0$  itself with the transverse and traceless (TT) gauge condition.

On the other hand, it was known that the operator approach (method) [7] is very useful to derive the quasinormal modes of spin-2 fields in the non-rotating BTZ black hole background in the framework of cosmological topologically massive gravity. This method has been applied to new massive gravity to derive their quasinormal modes of the non-rotating BTZ black hole [8].

In this work, we obtain quasinormal modes of the non-rotating BTZ black hole in spin-3 topologically massive gravity by directly solving the first-order equation with the TT gauge condition in the operator approach. This method shows clearly how to derive quasinormal modes without sign ambiguity in mass.

## 2 Perturbation analysis for spin-3 fields

Since the spin-3 fluctuations on  $\text{AdS}_3$  or BTZ background was formulated in [3], let us write down the perturbation equation for the spin-3 fields  $\Phi_{\mu\nu\lambda} = e_{\mu ab}\bar{e}_\nu^a\bar{e}_\lambda^b$  with  $e_{\mu ab}$  spin-3

connection and  $\bar{e}_\nu^a$  the background dreibein as

$$\bar{\square}\Phi^{\rho\alpha\beta} + \frac{1}{2\mu}\epsilon^{\rho\mu\nu}\bar{\nabla}_\mu\bar{\square}\Phi_\nu^{\alpha\beta} = 0. \quad (1)$$

Similar to the perturbed equation of the (spin-2) graviton,

$$(\bar{\square} + \frac{2}{l^2})h^\rho{}_\sigma + \frac{1}{\mu}\epsilon^{\rho\mu\nu}\bar{\nabla}_\mu(\bar{\square} + \frac{2}{l^2})h_{\nu\sigma} = 0, \quad (2)$$

the spin-3 fluctuation also satisfies a third-order differential equation.

In this work, we consider the non-rotating BTZ black hole with the mass  $M = 1$  and the  $\text{AdS}_3$  curvature radius  $\ell = 1$  in global coordinates as

$$ds_{\text{BTZ}}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -\sinh^2\rho d\tau^2 + \cosh^2\rho d\phi^2 + d\rho^2, \quad (3)$$

where the event horizon is located at  $\rho = 0$ , while the infinity is at  $\rho = \infty$ . Here we note that  $\bar{g}_{\mu\nu} = \bar{e}_\mu^a \bar{e}_\nu^b \eta_{ab}$ . In terms of the light-cone coordinates  $u/v = \tau \pm \phi$ , the metric tensor  $\bar{g}_{\mu\nu}$  takes the form of

$$\bar{g}_{\mu\nu} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\cosh 2\rho & 0 \\ -\frac{1}{4}\cosh 2\rho & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

Then the metric tensor (4) admits the Killing vector fields  $L_k$  ( $k = 0, -1, 1$ ) for the local  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  algebra as

$$L_0 = -\partial_u, \quad L_{-1/1} = e^{\mp u} \left[ -\frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v \mp \frac{1}{2} \partial_\rho \right], \quad (5)$$

and  $\bar{L}_0$  and  $\bar{L}_{-1/1}$  are obtained by substituting  $u \leftrightarrow v$ . Locally, they form a basis of the  $\text{SL}(2, \mathbb{R})$  Lie algebra as

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0. \quad (6)$$

In the BTZ black hole background, the spin-3 field of  $\Phi^{\rho\mu\nu}$  determined by a third-order differential equation (1) is totally symmetric and satisfies the TT gauge condition

$$\Phi_\mu^{\mu\nu} = 0, \quad \bar{\nabla}^\mu \Phi_{\mu\nu\rho} = 0. \quad (7)$$

Hence its number of propagating degrees of freedom is counted to be one as

$$10 - 3 - 6 = 1, \quad (8)$$

which corresponds to a single massive propagating mode as that in  $\text{AdS}_3$  background [3]. The third-order equation (1) can also be expressed as

$$(\mathcal{D}^M \mathcal{D}^L \mathcal{D}^R \Phi)^{\rho\mu\nu} = 0 \quad (9)$$

in terms of mutually commuting operators of

$$(\mathcal{D}^{L/R})^{\rho\nu} = \delta^{\rho\nu} \pm \frac{1}{2}\epsilon^{\rho\mu\nu}\bar{\nabla}_\mu, \quad (\mathcal{D}^M)^{\rho\nu} = \delta^{\rho\nu} + \frac{1}{2\mu}\epsilon^{\rho\mu\nu}\bar{\nabla}_\mu. \quad (10)$$

At the critical point of  $\mu = 1$ , the operators  $\mathcal{D}^M$  and  $\mathcal{D}^L$  degenerate. We note that Eq. (9) is reduced to Eq. (1) when using the BTZ background

$$\bar{R}_{\rho\sigma\mu\nu} = -(\bar{g}_{\rho\mu}\bar{g}_{\sigma\nu} - \bar{g}_{\rho\nu}\bar{g}_{\sigma\mu}), \quad \bar{R}_{\mu\nu} = -2\bar{g}_{\mu\nu}, \quad (11)$$

together with the TT gauge condition and the relation of  $[\bar{\nabla}_\mu, \bar{\nabla}_\nu]\Phi^{\mu\alpha\beta} = -4\Phi_\nu^{\alpha\beta}$ . Therefore, the third-order equation (1) can be decomposed into three first-order differential equations:

$$(\mathcal{D}^M\Phi)^{\rho\mu\nu} = 0, \quad (\mathcal{D}^L\Phi)^{\rho\mu\nu} = 0, \quad (\mathcal{D}^R\Phi)^{\rho\mu\nu} = 0, \quad (12)$$

for a massive, a left-moving, and a right-moving degree of freedom, respectively.

Importantly, three first-order differential equations (12) can be simply rewritten in terms of a single massive first-order differential equation as

$$\epsilon_\rho^{\alpha\beta}\bar{\nabla}_\alpha\Phi_{\beta\mu\nu} + m\Phi_{\rho\mu\nu} = 0 \quad (13)$$

with  $m = 2\mu$ , 2, and  $-2$ . On the other hand, it could also be expressed in terms of a second-order differential equation [6] as

$$\left[\bar{\square}^2 - m^2 + 4\right]\Phi_{\rho\mu\nu} = 0. \quad (14)$$

At this stage, we wish to point out the presence of sign ambiguity  $\pm m$  in the second-order equation (14). In order to avoid this ambiguity, one could directly solve the first-order equation (13) with the TT gauge condition.

Having the structure in mind, let us find quasinormal modes for the spin-3 field in the BTZ background by solving the equation of motion (13) with the TT gauge condition. In order to implement the operator method [7, 8], let us choose either the anti-chiral highest weight condition of  $L_1\Phi_{\rho\mu\nu} = 0$  or the chiral highest weight condition of  $\bar{L}_1\Phi_{\rho\mu\nu} = 0$ , but not both simultaneously. Actually, we note that for a generic symmetric tensor  $\Phi_{\rho\mu\nu}$ , the transversality condition of  $\bar{\nabla}^\mu\Phi_{\mu\nu\rho} = 0$  is not equivalent to choosing the chiral (anti-chiral) highest weight condition.

### 3 Left-moving quasinormal modes

The least damped ( $n = 0$ ) quasinormal mode can be found by considering the form

$$\Phi_{\rho\mu\nu}(u, v, \rho) = e^{-i\omega\tau - ik\phi}F_{\rho\mu\nu}(\rho) = e^{-ihu - i\bar{h}v}F_{\rho\mu\nu}(\rho) \quad (15)$$

with  $\omega = h + \bar{h}$  and  $k = h - \bar{h}$ . This is the primary field which satisfies

$$L_0 \Phi_{\rho\mu\nu}(u, v, \rho) = i h \Phi_{\rho\mu\nu}(u, v, \rho), \quad \bar{L}_0 \Phi_{\rho\mu\nu}(u, v, \rho) = i \bar{h} \Phi_{\rho\mu\nu}(u, v, \rho). \quad (16)$$

Note here that the subscript  $\rho$  in  $\Phi_{\rho\mu\nu}(u, v, \rho)$  is a dummy index, while  $\rho$  in the argument is the radial coordinate in (3). It seems to be a formidable task to solve the first-order equation with the TT gauge condition without choosing a simplified form of  $F_{\rho\mu\nu}$ . Inspired by the lesson learned from the spin-2 analysis [7, 8], after tedious computations, we find the explicit solution

$$F_{u\mu\nu}(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

$$F_{v\mu\nu}(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh 2\rho} \\ 0 & \frac{2}{\sinh 2\rho} & \frac{4}{\sinh^2 2\rho} \end{pmatrix} F_{vvv}(\rho), \quad (18)$$

$$F_{\rho\mu\nu}(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{\sinh 2\rho} & \frac{4}{\sinh^2 2\rho} \\ 0 & \frac{4}{\sinh^2 2\rho} & \frac{8}{\sinh^3 2\rho} \end{pmatrix} F_{vvv}(\rho), \quad (19)$$

which imply that  $F_{vvv}(\rho)$  is a single massive propagating mode. Here Eq. (18) is similar to the spin-2 case [7], while Eqs. (17) and (19) represent new features of the spin-3 field.

Under the form of  $\Phi_{\rho\mu\nu}(u, v, \rho)$  in Eq. (15) with  $F_{\rho\mu\nu}(\rho)$  in Eqs. (17)-(19), the transversality condition of  $\bar{\nabla}^\mu \Phi_{\mu\nu\rho} = 0$  is equivalent to the anti-chiral highest weight condition of  $L_1 \Phi_{\rho\mu\nu} = 0$ , giving the constraint

$$\sinh 2\rho \left[ \frac{d}{d\rho} F_{vvv}(\rho) \right] + 2i(\bar{h} + h \cosh 2\rho) F_{vvv}(\rho) = 0. \quad (20)$$

We emphasize that Eq. (20) takes the form of the equation for the scalar field  $F_{vvv}(\rho)$ , not a third-rank tensor field. Then, its solution is given by

$$F_{vvv}(\rho) = C(\sinh 2\rho)^{-ih} (\tanh \rho)^{-i\bar{h}} \quad (21)$$

with a constant  $C$ . Finally, the equation of motion (13) determines  $h$  as a function of  $m$  of

$$h = -ih_L(m), \quad h_L(m) = \frac{1}{2}(m-2). \quad (22)$$

Thus, the solution is summarized as

$$\Phi_{\rho\mu\nu}^L(u, v, \rho) = e^{ik(\tau-\phi)-2h_L(m)\tau} F_{\rho\mu\nu}^L(\rho) \quad (23)$$

where  $F_{\rho\mu\nu}^L(\rho)$  is given by Eqs. (17)-(19) with

$$F_{vvv}^L(\rho) = (\sinh \rho)^{-2h_L(m)} (\tanh \rho)^{ik}. \quad (24)$$

Considering the form of quasinormal frequency

$$\omega = \omega_{\text{Re}} - i\omega_{\text{Im}}, \quad (25)$$

we read off it from Eq. (23)

$$\omega_L = -k - 2ih_L(m). \quad (26)$$

Thus, the solution (23) corresponds to a left-moving massive quasinormal mode of the least damped ( $n = 0$ ) case for  $m = 2\mu$ , leading to

$$h_L(\mu) = \mu - 1 > 0 \quad \mu > 1. \quad (27)$$

As is expected by the anti-chiral gravity, we observe that there is no quasinormal modes ( $h_L = 0$ ) at the anti-chiral point of  $\mu = 1$ . Also we observe that the asymptotic  $\rho$ -dependence of  $F_{vvv}^L$  takes the form of  $F_{vvv}^L \sim e^{2(1-\mu)\rho}$ , which is compared to the spin-2 asymptotic dependence of  $F_{vv} \sim e^{(1-\mu)\rho}$  for  $m = \mu$  [9].

In order to derive the higher-order quasinormal modes, we act on the anti-chiral highest weight quasinormal modes with the operator of  $\bar{L}_{-1}L_{-1}$ . The effect of this will be to replace  $\omega_{\text{Im}}$  by  $\omega_{\text{Im}} + 2$  in Eq. (23). Hence one could expect to have

$$\Phi_{\rho\mu\nu}^{(n)L}(u, v, \rho) = \left( \bar{L}_{-1}L_{-1} \right)^n \Phi_{\rho\mu\nu}^L(u, v, \rho), \quad (28)$$

which are descendents of  $\Phi_{\rho\mu\nu}^L(u, v, \rho)$ . Since  $\bar{L}_{-1}L_{-1}$  commutes with the equation (13),  $\Phi_{\rho\mu\nu}^{(n)L}(u, v, \rho)$  is again the solution to the first-order equation with the same boundary condition of asymptotic fall-off as in  $\Phi_{\rho\mu\nu}^L(u, v, \rho)$ . Hence, the complete tower of the left-moving spin-3 quasinormal modes could be generated from  $\Phi_{\rho\mu\nu}^L(u, v, \rho)$ .

Consequently, the corresponding quasinormal frequencies are given by

$$\omega_L^n = -k - 2i\left(h_L(\mu) + n\right), \quad n \in \mathbb{Z}. \quad (29)$$

## 4 Right-moving quasinormal modes

On the other hand, right-moving quasinormal modes of the least damped case can be obtained by substitution of  $u \rightarrow v$ ,  $h \rightarrow \bar{h}$ , and  $m \rightarrow -m$ . Explicitly, they take the form of

$$F_{u\mu\nu}(\rho) = \begin{pmatrix} 1 & 0 & \frac{2}{\sinh 2\rho} \\ 0 & 0 & 0 \\ \frac{2}{\sinh 2\rho} & 0 & \frac{4}{\sinh^2 2\rho} \end{pmatrix} F_{uuu}(\rho), \quad (30)$$

$$F_{v\mu\nu}(\rho) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (31)$$

$$F_{\rho\mu\nu}(\rho) = \begin{pmatrix} \frac{2}{\sinh 2\rho} & 0 & \frac{4}{\sinh^2 2\rho} \\ 0 & 0 & 0 \\ \frac{4}{\sinh^2 2\rho} & 0 & \frac{8}{\sinh^3 2\rho} \end{pmatrix} F_{uuu}(\rho), \quad (32)$$

which imply that  $F_{uuu}(\rho)$  is a single massive propagating mode. Here Eq. (30) is similar to the spin-2 case [7], while Eqs. (31) and (32) represent new features of the spin-3 field.

The transversality condition of  $\bar{\nabla}^\mu \Phi_{\mu\nu\rho} = 0$  in this case is now compatible with the chiral highest weight condition of  $\bar{L}_1 \Phi_{\rho\mu\nu} = 0$ , giving the differential equation of  $F_{uuu}(\rho)$

$$\sinh 2\rho \left[ \frac{d}{d\rho} F_{uuu}(\rho) \right] + 2i(h + \bar{h} \cosh 2\rho) F_{uuu}(\rho) = 0. \quad (33)$$

The solution is given by

$$F_{uuu}(\rho) = D(\sinh 2\rho)^{-i\bar{h}} (\tanh \rho)^{-ih} \quad (34)$$

with a constant  $D$ . The equation of motion (13) determines  $\bar{h}$  as a function of  $m$

$$\bar{h} = -ih_R(m), \quad h_R(m) = \frac{1}{2}(m - 2). \quad (35)$$

Considering  $m = -2\mu$ , one finds

$$h_R(\mu) = -\mu - 1 \equiv \tilde{\mu} - 1 > 0, \quad \mu < -1 \quad (\tilde{\mu} > 1). \quad (36)$$

Then, the  $n = 0$  least damped right-moving solution is given by

$$\Phi_{\rho\mu\nu}^R(u, v, \rho) = e^{-ik(\tau+\phi)-2h_R(\mu)\tau} F_{\rho\mu\nu}^R(\rho) \quad (37)$$

where  $F_{\rho\mu\nu}^R(\rho)$  is given by Eqs. (30)-(32) with

$$F_{uuu}^R(\rho) = (\sinh \rho)^{-2h_R(m)} (\tanh \rho)^{-ik}. \quad (38)$$

Thus, its quasinormal mode can be read off as

$$\omega_R = k - 2ih_R(\mu). \quad (39)$$

As is expected by the chiral gravity, we observe that there is no quasinormal modes ( $h_R = 0$ ) at the chiral point of  $\mu = -1$ .

Similarly, the higher-order quasinormal modes are obtained by acting the operator of  $\bar{L}_{-1}L_{-1}$  as

$$\Phi_{\rho\mu\nu}^{(n)R}(u, v, \rho) = \left(\bar{L}_{-1}L_{-1}\right)^n \Phi_{\rho\mu\nu}^R(u, v, \rho), \quad (40)$$

which are descendants of  $\Phi_{\rho\mu\nu}^R(u, v, \rho)$ . Its quasinormal frequencies are given by

$$\omega_R^n = k - 2i\left(h_R(\mu) + n\right). \quad (41)$$

In Table 1, we have briefly summarized the results by comparing the spin-2 field in Ref. [7, 8] with the spin-3 topologically massive gravity. For the spin-2 field satisfying (2), the QNMs for the left-moving component exist for only  $\mu > 1$ , while the QNMs for the right-moving one for only  $\mu < -1$  as in Ref. [7, 8]. Especially, the result of Ref. [7] is obtained by  $u \rightarrow v$ ,  $h \rightarrow \bar{h}$ , but not by  $m \rightarrow -m$ . Instead, the authors gave the mass ranges for the QNMs as the left-moving (right-moving) component for  $\mu > 1$  ( $\mu < -1$ ), which are exactly the same with replacing  $m$  by  $-m$  in the equation of motion as shown in Table. 1. The QNMs for the left-moving (right-moving) spin-3 field are, by the same token, valid for only  $\mu > 1$  ( $\mu < -1$ ). Here we also note that there are no QNMs at  $\mu = 1$  ( $\mu = -1$ ) for the left-moving (right-moving) spin-2 field, while at  $\mu = 1$  ( $\mu = -1$ ) for the left-moving (right-moving) spin-3 field, expected by anti-chiral (chiral) gravity, respectively.

## 5 Discussions

We have obtained quasinormal modes of BTZ black hole in spin-3 topologically massive gravity by directly solving the first-order equation with the transverse-traceless condition in the operator approach. We have found that there is no  $n = 0$  quasinormal modes ( $h_{L/R} = 0$ ) at the anti-chiral/chiral point of  $\mu = \pm 1$ .

It seemed that these are different from those with  $T_{L/R} = \frac{1}{2\pi}$  [6]

$$\omega_{sL}^n = k - 2\pi T_L i(2n + m + 1 - s), \quad \omega_{sR}^n = -k - 2\pi T_R i(2n + m + 1 + s), \quad (42)$$

which are obtained when solving the second-order differential equations for the  $s$ -rank tensor of spin- $s$  field imposed by the boundary conditions. We note that the signs  $\pm$  of real part



Solutions of first-order differential equations	
spin-2 field	
$\epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} + m h_{\mu\nu} = 0$	$\epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} - m h_{\mu\nu} = 0$
$L_1 h_{\mu\nu} = 0$	$\bar{L}_1 h_{\mu\nu} = 0$
$h_L(m) = \frac{m}{2} - \frac{1}{2} \Big _{m=\mu} = \frac{\mu}{2} - \frac{1}{2}$	$h_R(m) = \frac{m}{2} - \frac{1}{2} \Big _{m=-\mu} = -\frac{\mu}{2} - \frac{1}{2}$
$\omega_n^L = -k - 2i(h_L(m) + n)$	$\omega_n^R = k - 2i(h_R(m) + n)$
spin-3 field	
$\epsilon_\rho^{\alpha\beta} \bar{\nabla}_\alpha \Phi_{\beta\mu\nu} + m \Phi_{\rho\mu\nu} = 0$	$\epsilon_\rho^{\alpha\beta} \bar{\nabla}_\alpha \Phi_{\beta\mu\nu} - m \Phi_{\rho\mu\nu} = 0$
$L_1 \Phi_{\rho\mu\nu} = 0$	$\bar{L}_1 \Phi_{\rho\mu\nu} = 0$
$h_L(m) = \frac{m}{2} - 1 \Big _{m=2\mu} = \mu - 1$	$h_R(m) = \frac{m}{2} - 1 \Big _{m=-2\mu} = -\mu - 1$
$\omega_n^L = -k - 2i(h_L(m) + n)$	$\omega_n^R = k - 2i(h_R(m) + n)$

Table 1: Summary of the QNMs by comparing the spin-2 field in Ref. [8] with the spin-3 field in topologically massive gravity: The right-moving solution is obtained by solving the first-order equation of motion with the replacement of  $u \rightarrow v$ ,  $h \rightarrow \bar{h}$ , and  $m \rightarrow -m$ . The equation of motion with  $m$  allows only the left-moving (anti-chiral) solution, while the one with  $-m$  gives only the right-moving (chiral) solution in Ref. [8].

$\omega_{sL/R}^n$  are different from  $\mp$  of  $\omega_{L/R}^n$ . We adhere to the convention of Ref. [7] for the left/right-moving modes. Comparing the left-moving QNMs in Eq. (29) with  $\omega_{3L}^n$  with  $m = 2\mu$  leads to the same expression for the imaginary sector. The same thing happens for the spin-2 when comparing  $\omega_{2L}^n$  with  $m = \mu$ .

On the other hand, it seems that the right-moving QNMs in Eq. (41) are different from  $\omega_{3R}^n$ . However,  $\omega_{3R}^n$  could be recovered from the descendants of  $\Phi_{\rho\mu\nu}^R(u, v, \rho)$  [9]. That is, one has that  $\text{imaginary}[\omega_{3R}^n] = \text{imaginary}[\omega_R^{n+3}]$  for spin-3. Similarly, one has that  $\text{imaginary}[\omega_{2R}^n] = \text{imaginary}[\omega_R^{n+2}]$  for spin-2. We have constructed these descendants for the spin-2 case in Appendix A and the spin-3 case in Appendix B, whose asymptotic forms of relevant part are consistent with those obtained by solving the second-order differential equations.

For  $\mu = \pm 1$ , we expect to develop logarithmic modes of spin-3 field in the BTZ black hole background as did for spin-2 field [8, 10].

Consequently, the operator approach combined with the first-order equation is a useful method to derive QNMs of the BTZ black hole in the spin- $s$  topologically massive gravity. We suggest that two approaches to the left-moving QNMs are identical, while the right-moving QNMs of solving second-order equation are given by descendants of the operator approach.

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## Appendix: Descendants of spin- $s$ field

### A. Descendants of spin-2 field

It was known that by solving the first-order equation of  $\epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu} + m h_{\mu\nu} = 0$  with the TT condition, one has the ingoing highest weight solution for the left-moving spin-2 field near the horizon [8, 9]

$$h_{\mu\nu}^L = e^{(1-\mu)t+ik(t-\phi)} (\sinh \rho)^{1-\mu} (\tanh \rho)^{ik} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{2}{\sinh 2\rho} \\ 0 & \frac{2}{\sinh 2\rho} & \frac{4}{\sinh^2 2\rho} \end{pmatrix}. \quad (43)$$

On the other hand, by the substitution of  $u \rightarrow v$ ,  $h \rightarrow \bar{h}$ , and  $m \rightarrow -m$ , we have the ingoing highest weight solution for the right-moving spin-2 field near the horizon

$$h_{\mu\nu}^R = e^{(\mu+1)t-ik(t+\phi)} (\sinh \rho)^{\mu+1} (\tanh \rho)^{-ik} \begin{pmatrix} 1 & 0 & \frac{2}{\sinh 2\rho} \\ 0 & 0 & 0 \\ \frac{2}{\sinh 2\rho} & 0 & \frac{4}{\sinh^2 2\rho} \end{pmatrix}. \quad (44)$$

By acting the operator of  $\bar{L}_{-1}L_{-1}$ , the second descendent for the right-moving mode is given by

$$\begin{aligned} h_{\mu\nu}^{(2)R} &= \left( \bar{L}_{-1}L_{-1} \right)^2 h_{\mu\nu}^R \\ &= \frac{1}{8} e^{-2h_R(\mu)t-ik(t+\phi)} (\sinh \rho)^{\mu-3} (\tanh \rho)^{-ik} \begin{pmatrix} m_{uu} & 0 & \frac{m_{u\rho}}{\sinh 2\rho} \\ 0 & 0 & 0 \\ \frac{m_{u\rho}}{\sinh 2\rho} & 0 & \frac{m_{\rho\rho}}{\sinh^2 2\rho} \end{pmatrix}, \end{aligned} \quad (45)$$

where

$$h_R(\mu) = -\frac{\mu}{2} + \frac{3}{2}, \quad (46)$$

and

$$\begin{aligned} m_{uu} = & (\mu(1+\mu) - k^2 - ik(1+2\mu)) \times \\ & (16 - 13\mu + 3\mu^2 - 8k^2 + 4\mu(-3+\mu) \cosh 2\rho + \mu(1+\mu) \cosh 4\rho \\ & - 8ik(-3+\mu + \mu \cosh 2\rho)), \end{aligned} \quad (47)$$

$$\begin{aligned} m_{u\rho} = & \frac{1}{8 \cosh^4 \rho} \{ -16 + 40\mu + 139\mu^2 - 150\mu^3 + 35\mu^4 - k^2(172 - 475\mu + 243\mu^2) \\ & + 48k^4 - ik(100 + 283\mu - 465\mu^2 + 150\mu^3 + 16k^2(10 - 11\mu)) \\ & + 4(2(1-\mu)^2(-11 - 15\mu + 7\mu^2) - k^2(39 - 181\mu + 90\mu^2) + 16k^4 \\ & + ik(7 - 102\mu + 177\mu^2 - 58\mu^3 - 62k^2(1-\mu))) \cosh 2\rho \\ & + 4(-2 + 6\mu + 21\mu^2 - 26\mu^3 + 7\mu^4 - k^2(25 - 73\mu + 35\mu^2) + 4k^4 \\ & - ik(15 + 41\mu - 75\mu^2 + 26\mu^3 + 4k^2(6 - 5\mu))) \cosh 4\rho \\ & - 4(1-\mu)(2(1-2\mu^2 + \mu^3) + k^2(5 - 6\mu) - ik(1 - 9\mu + 6\mu^2 - 2k^2)) \cosh 6\rho \\ & + \mu(1-\mu)(\mu(1-\mu) + k^2 - ik(1-2\mu)) \cosh 8\rho \}, \end{aligned} \quad (48)$$

$$\begin{aligned} m_{\rho\rho} = & \frac{1}{4 \cosh^4 \rho} \{ 140 - 196\mu + 391\mu^2 - 210\mu^3 + 35\mu^4 - k^2(502 - 665\mu + 243\mu^2) \\ & + 48k^4 - ik(14 + 823\mu - 651\mu^2 + 150\mu^3 + 16k^2(14 - 11\mu)) \\ & - 8(28 + 44\mu - 77\mu^2 + 42\mu^3 - 7\mu^4 + k^2(62 - 133\mu + 45\mu^2) - 8k^4 \\ & - ik(86 - 153\mu + 129\mu^2 - 29\mu^3 - k^2(46 - 31\mu))) \cosh 2\rho \\ & + 4(28 - 44\mu + 75\mu^2 - 42\mu^3 + 7\mu^4 - k^2(82 - 121\mu + 35\mu^2) + 4k^4 \\ & + ik(2 - \mu)(1 - 71\mu + 26\mu^2 - 20k^2)) \cosh 4\rho \\ & - 8(2 - \mu)(2 + 3\mu - 4\mu^2 + \mu^3 + k^2(5 - 3\mu) - ik(5 - 9\mu + 3\mu^2 - k^2)) \cosh 6\rho \\ & + (1 - \mu)(2 + \mu)((1 - \mu)(2 - \mu) - k^2 + ik(3 - 2\mu)) \cosh 8\rho \}. \end{aligned} \quad (49)$$

From  $(\sinh \rho)^{(\mu-3)}$  in Eq. (45), its asymptotic form is given by

$$h_{\rho\rho}^{(2)R} \sim e^{(\mu-3)\rho}, \quad \rho \rightarrow \infty, \quad (50)$$

which is consistent with that of  $n = 0$  right-moving quasinormal modes obtained by solving the second-order differential equation [6]. Explicitly, from Eq. (3.55) in Ref. [6], we recover the same asymptotic form of  $R_{22}(\xi) \sim e^{(m-3)\xi}$  for  $\xi = \rho$  and  $m = \mu$ .

## B. Descendants of spin-3 field

As was done in the spin-2 field, from the right-moving highest weight solution for the spin-3 field in Eqs. (37) and (38), the third descendent quasinormal modes can be computed as

$$\begin{aligned}\Phi_{\rho\mu\nu}^{(3)R} &= \left(\bar{L}_{-1}L_{-1}\right)^3 \Phi_{\rho\mu\nu}^R \\ &= \frac{1}{8} e^{-2h_R(\mu)t - ik(t+\phi)} (\sinh \rho)^{2(\mu-2)} (\tanh \rho)^{-ik} F_{\rho\mu\nu}^{(3)}(\rho),\end{aligned}\tag{51}$$

where

$$h_R(\mu) = -\mu + 2,\tag{52}$$

and

$$\begin{aligned}F_{u\mu\nu}^{(3)}(\rho) &= \begin{pmatrix} m_{uuu} & 0 & \frac{m_{uu\rho}}{\sinh 2\rho} \\ 0 & 0 & 0 \\ \frac{m_{uu\rho}}{\sinh 2\rho} & 0 & \frac{m_{u\rho\rho}}{\sinh^2 2\rho} \end{pmatrix}, \\ F_{v\mu\nu}^{(3)}(\rho) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ F_{\rho\mu\nu}^{(3)}(\rho) &= \begin{pmatrix} \frac{m_{uu\rho}}{\sinh 2\rho} & 0 & \frac{m_{u\rho\rho}}{\sinh^2 2\rho} \\ 0 & 0 & 0 \\ \frac{m_{u\rho\rho}}{\sinh^2 2\rho} & 0 & \frac{m_{\rho\rho\rho}}{\sinh^3 2\rho} \end{pmatrix}\end{aligned}\tag{53}$$

with

$$\begin{aligned}m_{uuu} &= \{(1+2\mu)(4\mu(1+\mu) - 3k^2) - ik(2(1+6\mu+6\mu^2) - k^2)\} \times \\ &\quad \{2(-2+\mu)(12-19\mu+10\mu^2-12k^2) - 2ik(44-51\mu+18\mu^2-4k^2) \\ &\quad + 3\mu(29-33\mu+10\mu^2-8k^2-16ik(-2+\mu)) \cosh 2\rho \\ &\quad + 6\mu(-2+\mu-ik)(1+2\mu) \cosh 4\rho + \mu(1+\mu)(1+2\mu) \cosh 6\rho\}\end{aligned}\tag{54}$$

$$\begin{aligned}
m_{uu\rho} = & \frac{1}{64 \cosh^6 \rho} \{ 84(1-2\mu)^2(27+90\mu-25\mu^2-100\mu^3+44\mu^4) \\
& + 28k^2(115-912\mu-1485\mu^2+4560\mu^3-2172\mu^4) \\
& - 12(-6(9-42\mu-523\mu^2+700\mu^3+636\mu^4-1120\mu^5+352\mu^6) \\
& + 4k^2(-278+819\mu+1382\mu^2-4340\mu^3+2104\mu^4) \\
& + k^4(-214+2242\mu-2244\mu^2)+40k^6 \\
& + ik(-525-3181\mu+6468\mu^2+7512\mu^3-17080\mu^4+6528\mu^5 \\
& - k^2(477+1791\mu-8822\mu^2+5792\mu^3)+4k^4(-57+116\mu))) \cosh 2\rho \\
& - 3(-6(1-2\mu)^2(108+336\mu-169\mu^2-420\mu^3+220\mu^4) \\
& + 2k^2(-516+3296\mu+5585\mu^2-18560\mu^3+9532\mu^4) \\
& - 16k^4(29-282\mu+276\mu^2)+64k^6 \\
& + ik(576-6603\mu+13888\mu^2+14768\mu^3-37760\mu^4+15600\mu^5 \\
& - k^2(1168+3739\mu-18272\mu^2+12308\mu^3)+64k^4(-7+13\mu))) \cosh 4\rho \\
& - 2(-36+216\mu+2966\mu^2-5400\mu^3-2776\mu^4+8640\mu^5-3520\mu^6 \\
& + 4k^2(-236+843\mu+1302\mu^2-4620\mu^3+2664\mu^4) \\
& - 2k^4(158-951\mu+942\mu^2)+16k^6 \\
& + ik(-474-2723\mu+7188\mu^2+6152\mu^3-20040\mu^4+9600\mu^5 \\
& - k^2(498+2045\mu-8442\mu^2+6112\mu^3)+24k^4(-7+12\mu))) \cosh 6\rho \\
& + 6(-1+2\mu)(2(-9+77\mu^2-46\mu^3-116\mu^4+88\mu^5) \\
& + 4k^2(-46+36\mu+358\mu^2-404\mu^3)+4k^4(-4+11\mu) \\
& - ik(-15-175\mu-114\mu^2-468\mu^3+424\mu^4 \\
& + k^2(1+123\mu-190\mu^2)+4k^4)) \cosh 8\rho \\
& + 6\mu(-1+2\mu)(2\mu(5-10\mu-8\mu^2+16\mu^3)+k^2(4+16\mu-48\mu^2)+2k^4 \\
& + ik(-5+18\mu+28\mu^2-64\mu^3+k^2(-3+16\mu))) \cosh 10\rho \\
& + (2\mu^2(1-4\mu^2)(1-4\mu^2+3k^2)+ik\mu(1-4\mu^2)(1+12\mu^2-k^2)) \cosh 12\rho \}
\end{aligned} \tag{55}$$

$$\begin{aligned}
m_{u\rho\rho} = & \frac{1}{16 \cosh^6 \rho} \{ 2(12(189 + 284\mu - 518\mu^2 - 632\mu^3 + 2037\mu^4 - 1428\mu^5 + 308\mu^6) \\
& - k^2(1010 - 4695\mu + 37881\mu^2 - 45384\mu^3 + 15204\mu^4) \\
& + k^4(841 - 2991\mu + 2118\mu^2) - 80k^6) \\
& - 2ik(2(2931 - 2812\mu - 5415\mu^2 + 24814\mu^3 - 22050\mu^4 + 5796\mu^5) \\
& + k^2(210 - 12953\mu + 23325\mu^2 - 10678\mu^3) + 36k^4(-17 + 25\mu)) \\
& - 12(630 - 752\mu + 1790\mu^2 + 2108\mu^3 - 6848\mu^4 + 4848\mu^5 - 1056\mu^6 \\
& + k^2(-1167 - 1619\mu + 10438\mu^2 - 12608\mu^3 + 4208\mu^4) \\
& + k^4(-433 + 1643\mu - 1122\mu^2) + 20k^6 \\
& + ik(-779 - 1969\mu - 3124\mu^2 + 13816\mu^3 - 12360\mu^4 + 3264\mu^5 \\
& + k^2(397 - 3487\mu + 6434\mu^2 - 2896\mu^3) + 8k^4(-21 + 29\mu))) \cosh 2\rho \\
& - 3(-4(360 + 448\mu - 1075\mu^2 - 1198\mu^3 + 4033\mu^4 - 2940\mu^5 + 660\mu^6) \\
& + k^2(272 - 3999\mu + 23723\mu^2 - 28792\mu^3 + 9532\mu^4) \\
& - 16k^4(59 - 221\mu + 138\mu^2) + 32k^6 \\
& + ik(3536 - 4486\mu - 7450\mu^2 + 32004\mu^3 - 29100\mu^4 + 7800\mu^5 \\
& + k^2(496 - 7765\mu + 14255\mu^2 - 6154\mu^3) + 32k^4(-11 + 13\mu))) \cosh 4\rho \\
& - 2(810 - 1536\mu + 2434\mu^2 + 2628\mu^3 - 9728\mu^4 + 7440\mu^5 - 1760\mu^6 \\
& + k^2(-949 - 1677\mu + 13818\mu^2 - 16128\mu^3 + 5328\mu^4) \\
& + k^4(-599 + 1653\mu - 942\mu^2) + 8k^6 \\
& + ik(-1 + \mu)(717 + 2212\mu + 6256\mu^2 - 12600\mu^3 + 4800\mu^4 \\
& + k^2(-291 + 4318\mu - 3056\mu^2) + 144k^4)) \cosh 6\rho \\
& + 6(-1 + \mu)(4(-15 - 19\mu + 35\mu^2 + 83\mu^3 - 128\mu^4 + 44\mu^5) \\
& - k^2(46 + 321\mu - 788\mu^2 + 404\mu^3) + k^4(-34 + 44\mu) \\
& - ik(2(-59 + 101\mu + 272\mu^2 - 518\mu^3 + 212\mu^4) \\
& + k^2(-66 + 267\mu - 190\mu^2) + 4k^4)) \cosh 8\rho \\
& + 6(1 - 3\mu + 2\mu^2)(2(-3 + 7\mu + 4\mu^2 - 16\mu^3 + 8\mu^4) + k^2(-5 + 28\mu - 24\mu^2) + k^4 \\
& + ik(-5 - 14\mu + 52\mu^2 - 32\mu^3 + k^2(-5 + 8\mu))) \cosh 10\rho \\
& + (\mu(1 - 3\mu + 2\mu^2)(4\mu(1 - 3\mu + 2\mu^2) + 3k^2(1 - 2\mu) \\
& + ik(-2 + 12\mu - 12\mu^2 + k^2)) \cosh 12\rho \},
\end{aligned} \tag{56}$$

$$\begin{aligned}
m_{\rho\rho\rho} = & \frac{1}{16 \cosh^6 \rho} \{ 4(3(3168 + 1488\mu + 3737\mu^2 - 13896\mu^3 + 15848\mu^4 - 7392\mu^5 + 1232\mu^6) \\
& - k^2(9806 - 30354\mu + 74763\mu^2 - 58848\mu^3 + 15204\mu^4) \\
& + k^4(3362 - 7782\mu + 4236\mu^2) - 80k^6) \\
& - 2ik(32556 + 26819\mu - 125136\mu^2 + 194416\mu^3 - 114240\mu^4 + 23184\mu^5 \\
& + k^2(7920 - 51493\mu + 60576\mu^2 - 21356\mu^3) + 12k^4(-133 + 150\mu)) \\
& + 12(6(-1056 + 144\mu + 1003\mu^2 - 3972\mu^3 + 4516\mu^4 - 2112\mu^5 + 352\mu^6) \\
& - 4k^2(-575 - 4652\mu + 1043 + \mu^2 - 8268\mu^3 + 2104\mu^4) \\
& + 2k^4(878 - 2165\mu + 1122\mu^2) + 40k^6 \\
& - ik(-7452 + 5081\mu - 36124\mu^2 + 54952\mu^3 - 32360\mu^4 + 6528\mu^5 \\
& + k^2(3696 - 14033\mu + 16914\mu^2 - 5792\mu^3) + k^4(-444 + 464\mu))) \cosh 2\rho \\
& + 3(6(2139 + 498\mu + 2407\mu^2 - 9888\mu^3 + 11224\mu^4 - 5280\mu^5 + 880\mu^6) \\
& - 2k^2(5561 - 22802\mu + 49529\mu^2 - 39024\mu^3 + 9532\mu^4) \\
& + 16k^4(247 - 602\mu + 276\mu^2) - 64k^6 \\
& - ik(20475 + 13397\mu - 89648\mu^2 + 133648\mu^3 - 78640\mu^4 + 15600\mu^5 \\
& + k^2(6735 - 32439\mu + 38748\mu^2 - 12308\mu^3) + 64k^4(-15 + 13\mu))) \cosh 4\rho \\
& - 2(9486 - 5892\mu - 10870\mu^2 + 38952\mu^3 - 44776\mu^4 + 21120\mu^5 - 3520\mu^6 \\
& + 8k^2(8 - 3141\mu + 7788\mu^2 - 5754\mu^3 + 1332\mu^4) \\
& - 2k^4(1283 - 2355\mu + 942\mu^2) + 16k^6 \\
& + ik(-8913 + 15535\mu - 56124\mu^2 + 86072\mu^3 - 49560\mu^4 + 9600\mu^5 \\
& + k^2(4071 - 20363\mu + 21054\mu^2 - 6112\mu^3) + 24k^4(-17 + 12\mu))) \cosh 6\rho \\
& + 6(-3 + 2\mu)(2(-114 + 4\mu - 165\mu^2 + 538\mu^3 - 396\mu^4 + 88\mu^5) \\
& + k^2(80 - 956\mu + 1218\mu^2 - 404\mu^3) + k^4(-52 + 44\mu) \\
& - ik(-260 - 205\mu + 1714\mu^2 - 1604\mu^3 + 424\mu^4 \\
& + k^2(-184 + 411\mu - 190\mu^2) + 4k^4)) \cosh 8\rho \\
& + 6(3 - 5\mu + 2\mu^2)(2(-21 + 11\mu + 50\mu^2 - 56\mu^3 + 16\mu^4) - 8k^2(5 - 12\mu + 6\mu^2) \\
& + 2k^4 + ik(7 - 130\mu + 180\mu^2 - 64\mu^3 + k^2(-17 + 16\mu))) \cosh 10\rho \\
& + (3 - 11\mu + 12\mu^2 - 4\mu^3)(6 - 22\mu + 24\mu^2 - 8\mu^3 - 6k^2(1 - \mu) \\
& + ik(11 - 24\mu + 12\mu^2 - k^2)) \cosh 12\rho \}. \tag{57}
\end{aligned}$$

From  $(\sinh \rho)^{2(\mu-2)}$  in Eq. (51), its asymptotic form is given by

$$\Phi_{\rho\rho\rho}^{(3)R} \sim e^{2(\mu-2)\rho}, \tag{58}$$

which coincides with that of  $n = 0$  right-moving quasinormal modes obtained by solving the second-order differential equation [6]. Explicitly, from Eq. (B.32) in Ref. [6], we recover the same asymptotic form of  $R_{222}(\xi) \sim e^{2(\frac{m}{2}-2)\xi}$  for  $\xi = \rho$  and  $m = 2\mu$ .

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